Mathematica Moravica
Vol. **20:2** (2016), 123–129

New symmetric identities of Carlitz's generalized twisted q-Bernoulli polynomials under S_3

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Abstract. In this paper, the authors consider the Carlitz's generalized twisted q-Bernoulli polynomials attached to χ and investigate some novel symmetric identities for these polynomials arising from the p-adic q-integral on $\mathbb{Z}_p$ under S_3.

1. Introduction

Recently, symmetric properties of some well-known polynomials arising from p-adic q-integrals on $\mathbb{Z}_p$ have been investigated extensively by many mathematicians. For example, Araci *et al.* [3] investigated some new symmetric identities of q-Frobenius polynomials under S_5 which are associated with the fermionic p-adic q-integral over the p-adic numbers field. Kim *et al.* [5] derived some novel identities of symmetry for the Carlitz q-Bernoulli polynomials invariant under S_4. Dolgy *et al.* [6] gave some symmetric identities generalized Carlitz's q-Bernoulli polynomials of the first kind under S_3. Duran *et al.* [7] investigated some new symmetric identities of q-Genocchi polynomials arising from the q-Volkenborn integral on $\mathbb{Z}_p$ under S_4. Duran *et al.* [8] obtained some new symmetric identities of weighted q-Genocchi polynomials using q -Volkenborn integral on $\mathbb{Z}_p$ under S_4. Duran *et al.* [9] considered some new symmetric identities of Carlitz's twisted (h,q)-Euler polynomials derived from p-adic invariant integral on $\mathbb{Z}_p$ under S_n. Additionaly, Kim *et al.* [12] discovered new identities of symmetry for generalized q-Bernoulli polynomials of the second kind under the Dihedral group D_3. Furthermore, Kim [13] presented some novel identities of symmetry for Carlitz's-type q-Bernoulli polynomials using p-adic q-integral on $\mathbb{Z}_p$ under symmetric group of degree five.

2010 *Mathematics Subject Classification.* Primary: 05A19, 05A30; Secondary: 11S80, 11B68.

Key words and phrases. Symmetric identities; Carlitz's generalized twisted q-Bernoulli polynomials; p-adic q-integral on $\mathbb{Z}_p$; Invariant under S_3.

As is well known, the familiar Bernoulli polynomials, $B_n(x)$, are defined by means of the following Taylor series expansion about $t=0$:

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}=\frac{t}{e^t-1}e^{xt}, \qquad (|t|<2\pi). \tag{1}$$

Upon setting $x=0$ in the Eq. (1), we have $B_n(0):=B_n$ that is popularly known as n-th Bernoulli number (see, e.g., [1], [4], [5], [6], [10], [11], [12], [13], [14], [15]).

Let p be a fixed odd prime number. Throughout the present paper, $\mathbb{Z}_p$, $\mathbb{Q}$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic rational integers, the field of rational numbers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}=\{1,2,3,\cdots\}$ and $\mathbb{N}^*=\mathbb{N}\cup\{0\}$. The normalized absolute value according to the theory of p-adic analysis is given by $|p|_p=p^{-1}$. The notation "q" can be considered as an indeterminate, a complex number $q\in\mathbb{C}$ with $|q|<1$, or a p-adic number $q\in\mathbb{C}_p$ with $|q-1|_p<p^{-\frac{1}{p-1}}$ and $q^x=\exp(x\log q)$ for $|x|_p\leq 1$. The q-analogue of x is defined by $[x]_q=(1-q^x)/(1-q)$. Observe that $\lim_{q\to 1}[x]_q=x$ for any x with $|x|_p\leq 1$ in the p-adic case (for details, see [1, 3, 5-15]).

For

$$g\in UD(\mathbb{Z}_p)=\{g\,|g:\mathbb{Z}_p\to\mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the bosonic p-adic q-integral on $\mathbb{Z}_p$ of a function $g\in UD(\mathbb{Z}_p)$ is defined by Kim in [11]:

$$I_q(g)=\int_{\mathbb{Z}_p} g(x)\,d\mu_q(x)=\lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{x=0}^{p^N-1} g(x)\,q^x. \tag{2}$$

Thus, in view of the Eq. (2), we have

$$qI_q(g_1)-I_q(g)=(q-1)\,f(0)+\frac{q-1}{\log q}f'(0)$$

where $g_1(x)=g(x+1)$.

For $d\in\mathbb{N}$ with $(p,d)=1$, let

$$X:=X_d=\varprojlim_{n}\mathbb{Z}/dp^n\mathbb{Z} \ \text{ and } X_1=\mathbb{Z}_p,$$

$$X^*=\bigcup_{\substack{0<a<dp\\(a,p)=1}}(a+dp\mathbb{Z}_p)$$

and

$$a+dp^n\mathbb{Z}_p=\{x\in X \mid x\equiv a\,(\mathrm{mod}\,dp^n)\}$$

where $a\in\mathbb{Z}$ lies in $0\leq a<dp^n$. For further details, see [3, 5-15].

Note that

$$\int_X g(x)\,d\mu_q(x)=\int_{\mathbb{Z}_p} g(x)\,d\mu_q(x)\,,\text{ for } g\in UD(\mathbb{Z}_p)\,.$$

Let χ be a primitive Dirichlet's character with conductor $d \in \mathbb{N}$. Details on the Dirichlet's character χ can be found in [2].

Let $T_p = \bigcup_{N\geq 1} C_{p^N} = \lim_{N\to\infty} C_{p^N}$, where $C_{p^N} = \left\{\zeta : \zeta^{p^N} = 1\right\}$ is the cyclic group of order p^N. For $\zeta \in T_p$, we indicate by $\phi_\zeta : \mathbb{Z}_p \to C_p$ the locally constant function $x \to \zeta^x$. For $q \in C_p$ with $|1-q|_p < 1$ and $\zeta \in T_p$, the Carlitz's generalized twisted q-Bernoulli polynomials attached to χ with Witt's formula are defined by the following p-adic q-integral on $\mathbb{Z}_p$, with respect to μ_q, in [14]:

$$\int_{\mathbb{Z}_p} \chi(y)\,\zeta^y\,[x+y]_q^n\,d\mu_q(y) = \beta_{n,\chi,q,\zeta}(x) \quad (n \geq 0)\,. \tag{3}$$

Substituting $x = 0$ into the Eq. (3) gives $\beta_{n,\chi,q,\zeta}(0) := \beta_{n,\chi,q,\zeta}$ that are called n-th Carlitz's generalized twisted q-Bernoulli number attached to χ.

By using the integral p-adic q-integral, we have the following relation:

$$\beta_{n,\chi,q,\zeta}(x) = [d]_q^{n-1}\sum_{i=0}^{d-1}\chi(i)\,q^i\zeta^i\beta_{n,\chi,q,\zeta}\Big(\frac{x+i}{d}\Big).$$

In the next section, we consider the Carlitz's generalized twisted q-Bernoulli polynomials attached to χ and investigate some novel symmetric identities for these polynomials arising from the p-adic q-integral on $\mathbb{Z}_p$ under symmetric group of degree three denoted by S_3. Further, in Corollary, we discuss some special cases of our results in this paper.

2. Novel identities of symmetry for $\beta_{n,\chi,q,\zeta}(x)$ under S_3

Let $\zeta \in T_p$, $q \in \mathbb{C}_p$ with $|q-1|_p < 1$, $w_i \in \mathbb{N}$ with $i \in \{1,2,3\}$ and χ be the trivial character. By the Eqs. (2) and (3), we discover

$$\int_{\mathbb{Z}_p}\chi(y)\,\zeta^{w_2w_3y}e^{[w_2w_3y+w_1w_2w_3x+w_1w_3i+w_1w_2j]_qt}d\mu_{q^{w_2w_3}}(y)$$

$$= \lim_{N\to\infty}\frac{1}{[dp^N]_{q^{w_2w_3}}}\sum_{y=0}^{dp^N-1}\chi(y)\,\zeta^{w_2w_3y}q^{w_2w_3y}e^{[w_2w_3y+w_1w_2w_3x+w_1w_3i+w_1w_2j]_qt}$$

$$= \lim_{N\to\infty}\frac{1}{[w_1dp^N]_{q^{w_1w_2w_3}}}\sum_{l=0}^{dw_1-1}\sum_{y=0}^{p^N-1}\zeta^{w_2w_3(l+w_1dy)}q^{w_2w_3(l+w_1dy)}\chi(l)$$
$$\times e^{[w_2w_3(l+w_1dy)+w_1w_2w_3x+w_1w_3i+w_1w_2j]_qt},$$

which yields

$$I = \frac{1}{[w_2w_3]_q}\sum_{i=0}^{dw_2-1}\sum_{j=0}^{dw_3-1}\chi(i)\,\chi(j)\,\zeta^{w_1w_3i+w_1w_2j}q^{w_1w_3i+w_1w_2j} \tag{4}$$

$$\times \int_{\mathbb{Z}_p} \chi(y)\, \zeta^{w_2w_3y} e^{[w_2w_3y+w_1w_2w_3x+w_1w_3i+w_1w_2j]_q t} d\mu_{q^{w_2w_3}}(y)$$

$$= \lim_{N\to\infty} \frac{1}{[w_1w_2w_3p^N]_q} \sum_{i=0}^{dw_2-1} \sum_{j=0}^{dw_3-1} \sum_{l=0}^{dw_1-1} \sum_{y=0}^{p^N-1} \chi(ijl)$$
$$\times \zeta^{w_2w_3(l+w_1dy)+w_1w_3i+w_1w_2j} q^{w_2w_3(l+w_1dy)+w_1w_3i+w_1w_2j}$$
$$\times e^{[w_2w_3(l+w_1dy)+w_1w_2w_3x+w_1w_3i+w_1w_2j]_q t}.$$

Note that Eq. (4) is invariant for any permutation $\sigma \in S_3$. Therefore, we acquire the following theorem.

Theorem 2.1. *Let $\zeta \in T_p$, $q \in \mathbb{C}_p$ with $|q-1|_p < 1$, $w_i \in \mathbb{N}$ with $i \in \{1,2,3\}$ and χ be the trivial character. Then the following expression*

$$I = \frac{1}{[w_{\sigma(2)}w_{\sigma(3)}]_q} \sum_{i=0}^{dw_{\sigma(2)}-1} \sum_{j=0}^{dw_{\sigma(3)}-1} \chi(i)\,\chi(j)\, \zeta^{w_{\sigma(1)}w_{\sigma(3)}i+w_{\sigma(1)}w_{\sigma(2)}j}$$
$$\times q^{w_{\sigma(1)}w_{\sigma(3)}i+w_{\sigma(1)}w_{\sigma(2)}j} \int_{\mathbb{Z}_p} \chi(y)\, \zeta^{w_{\sigma(2)}w_{\sigma(3)}y}$$
$$\times e^{[w_{\sigma(2)}w_{\sigma(3)}y+w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}x+w_{\sigma(1)}w_{\sigma(3)}i+w_{\sigma(1)}w_{\sigma(2)}j]_q t} d\mu_{q^{w_{\sigma(2)}w_{\sigma(3)}}}(y)$$

holds true for any $\sigma \in S_3$.

By using the definition of q-number, we obtain

(5)

$$\int_{\mathbb{Z}_p} \chi(y)\, \zeta^{w_2w_3y} e^{[w_2w_3y+w_1w_2w_3x+w_1w_3i+w_1w_2j]_q t} d\mu_{q^{w_2w_3}}(y)$$
$$= \sum_{n=0}^{\infty} [w_2w_3]_q^n \left(\int_{\mathbb{Z}_p} \chi(y)\, \zeta^{w_2w_3y} \left[y + w_1x + \frac{w_1}{w_2}i + \frac{w_1}{w_3}j \right]_{q^{w_2w_3}}^n d\mu_{q^{w_2w_3}}(y) \right) \frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty} [w_2w_3]_q^n \beta_{n,\chi,q^{w_2w_3},\zeta^{w_2w_3}} \left(w_1x + \frac{w_1}{w_2}i + \frac{w_1}{w_3}j \right) \frac{t^n}{n!}.$$

Thus, from Theorem 2.1 and Eq. (5), we state the following theorem.

Theorem 2.2. *Let $\zeta \in T_p$, $q \in \mathbb{C}_p$ with $|q-1|_p < 1$, $w_i \in \mathbb{N}$ with $i \in \{1,2,3\}$ and χ be the trivial character. For $n \geq 0$, the following*

$$I = [w_{\sigma(2)}w_{\sigma(3)}]_q^{n-1} \sum_{i=0}^{dw_{\sigma(2)}-1} \sum_{j=0}^{dw_{\sigma(3)}-1} \chi(i)\,\chi(j)\, \zeta^{w_{\sigma(1)}w_{\sigma(3)}i+w_{\sigma(1)}w_{\sigma(2)}j}$$
$$\times q^{w_{\sigma(1)}w_{\sigma(3)}i+w_{\sigma(1)}w_{\sigma(2)}j} \beta_{n,\chi,q^{w_{\sigma(2)}w_{\sigma(3)}},\zeta^{w_{\sigma(2)}w_{\sigma(3)}}} \left(w_{\sigma(1)}x + \frac{w_{\sigma(1)}}{w_{\sigma(2)}}i + \frac{w_{\sigma(1)}}{w_{\sigma(3)}}j \right)$$

holds true for any $\sigma \in S_3$.

Using binomial theorem and the definitions of q-number, we derive that

$$\begin{aligned}&\left[y+w_1x+\frac{w_1}{w_2}i+\frac{w_1}{w_3}j\right]^n_{q^{w_2w_3}}\\ = &\sum_{m=0}^{n}\binom{n}{m}\left(\frac{[w_1]_q}{[w_2w_3]_q}\right)^{n-m}[w_3i+w_2j]^{n-m}_{q^{w_1}}\,q^{m(w_1w_3i+w_1w_2j)}\,[y+w_1x]^m_{q^{w_2w_3}}\,,\end{aligned}$$

which gives

$$\begin{aligned}(6)\qquad &[w_2w_3]_q^{n-1}\sum_{i=0}^{dw_2-1}\sum_{j=0}^{dw_3-1}\chi(i)\,\chi(j)\,\zeta^{w_1w_3i+w_1w_2j}q^{w_1w_3i+w_1w_2j}\\ &\times\int_{\mathbb{Z}_p}\chi(y)\,\zeta^{w_2w_3y}\left[y+w_1x+\frac{w_1}{w_2}i+\frac{w_1}{w_3}j\right]^n_{q^{w_2w_3}}d\mu_{q^{w_2w_3}}(y)\\ =&\sum_{m=0}^{n}\binom{n}{m}[w_2w_3]_q^{m-1}[w_1]_q^{n-m}\,\beta_{m,\chi,q^{w_2w_3},\zeta^{w_2w_3}}(w_1x)\,\widehat{C}_{n,m,q^{w_1},\zeta^{w_1}}(w_2,w_3\mid\chi),\end{aligned}$$

where

$$\begin{aligned}(7)\qquad &\widehat{C}_{n,m,q,\zeta}(w_2,w_3\mid\chi)\\ &=\sum_{i=0}^{w_2-1}\sum_{j=0}^{w_3-1}\chi(i)\,\chi(j)\,\zeta^{w_3i+w_2j}q^{(m+1)(w_3i+w_2j)}\,[w_3i+w_2j]_q^{n-m}\,.\end{aligned}$$

Therefore, by Eq. (7), we arrive at the following theorem.

Theorem 2.3. *Let $\zeta\in T_p$, $q\in\mathbb{C}_p$ with $|q-1|_p<1$, $w_i\in\mathbb{N}$ with $i\in\{1,2,3\}$ and χ be the trivial character. For $n\geq 0$, the following expression*

$$\begin{aligned}&\sum_{m=0}^{n}\binom{n}{m}\left[w_{\sigma(2)}w_{\sigma(3)}\right]_q^{m-1}\left[w_{\sigma(1)}\right]_q^{n-m}\\ &\times\beta_{m,\chi,q^{w_{\sigma(2)}w_{\sigma(3)}},\zeta^{w_{\sigma(2)}w_{\sigma(3)}}}\left(w_{\sigma(1)}x\right)\widehat{C}_{n,m,q^{w_{\sigma(1)}},\zeta^{w_{\sigma(1)}}}(w_{\sigma(2)},w_{\sigma(3)}\mid\chi)\end{aligned}$$

holds true for some $\sigma\in S_3$.

3. Conclusion

In this study, we have investigated not only new but also interesting symmetric identities for Carlitz's generalized twisted q-Bernoulli polynomials attached to χ arising from the p-adic q-integral on $\mathbb{Z}_p$ under S_3. We note that for $\chi=1$, all our results in this paper reduce to the results of Carlitz's twisted q-Bernoulli polynomials under S_3. Moreover, in the case $\zeta=1$, all our results in this paper reduce to the results in [6].

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